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THE INFLUENCE OF INTERFACIAL PRESSURE FORCES ON THE CHARACTER OF TWO-PHASE FLOW MODEL EQUATIONS

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Abstract—Equations for two-phase flow are developed where the interfacial pressure p_i is the closure variable. The assumption that p_i is constant leads to variations of the single pressure model with several aphysical properties. Use of more realistic pressure distributions, for flow about solid particles, produces a model displaying added mass and drag effects and having real characteristic roots.

INTRODUCTION

In many applications involving the flow of two fluid phases, or of a fluid and solid phase, it is desirable to seek a dynamical description of the total flow in terms of quantities averaged in either space or time. In recent years there have been advanced formally exact derivations of these equations, for instance Ishii (1975). Like their turbulence counterparts, however, the difficulty in using such formulations lies in choosing appropriate closure forms for each of the many interaction terms introduced by the averaging process.

One peculiarity that has emerged for two-phase flow is the occurrence of complex characteristic roots in some model equation sets (Gidaspow 1974). The mathematical implications in this instance are unclear, but for linear partial differential equations initial-value problems would be ill-posed and their solutions can not be generated. No rigorous theorems are yet available, but Lyczkowski *et al.* (1975) have reported difficulty in obtaining numerical solutions to such equation sets. Several attempts have been made to improve the mathematical properties by introducing more physical properties into the description. Ramshaw & Trapp (1975) considered surface tension effects, while Travis *et al.* (1976) considered viscous stresses. This paper considers yet another phenomenon, interfacial pressure forces.

In addition, to produce the correct acceleration of small density phases during transient motion, the so-called added mass effect of the heavier fluid must be accounted for. Such effects do not arise naturally in most models and are either neglected or added in an *ad hoc* manner.

To help clarify some of these points, the present paper attempts to determine the consequences of various assumptions about the interfacial pressure distribution on the mathematical and physical character of the averaged, two-phase flow equations. The work starts with the formally exact equations of Ishii and through simplifying assumptions reduces them to a system for which closure depends only on the choice of the interfacial pressure. The properties of the final equations can then be understood in terms of the assumptions made regarding the small-scale flow near the interface.

The simplest choice for the interfacial pressure distribution, p_i, is that it is constant and equal to the bulk pressure. This leads to an equation set involving a single pressure and possessing real characteristics only when the two fluid velocities are equal. A slightly more general assumption, that the interface pressure is constant, p^*, but different from the bulk value, p, gives real values only if p^* is lower than p by an amount proportional to the square of the relative velocity. There is no obvious criteria, however, for choosing p^* and, in any event, the model equations do not display added mass behavior or drag forces.

We next use a theoretically and experimentally determined expression for the pressure distribution about a sphere to calculate the interfacial interaction term. The integrations are carried out in the limit of a large averaging volume and produce three new interaction terms characterized by three parameters: an added mass coefficient, μ; a dynamic pressure coefficient, ξ; and a drag parameter, η. These parameters depend on the nature of the microscopic flow and the statistical properties of the average, but indicate the general mathematical character of the pressure interaction.

The equations that result from using a realistic pressure distribution display added mass effects and interfacial drag and, most importantly, have real characteristic roots. Although the pressure distributions were obtained for isolated spherical particles, some justification is given for believing that similar results hold for all volume fractions and for other particle shapes.

It is concluded that the exact formulations of two-phase flow equations coupled with realistic and consistent assumptions concerning the small-scale flow properties can be helpful in producing model equations with reasonable mathematical and physical properties.

THE SIMPLIFIED MODEL EQUATIONS

We begin with the macroscopic equations for two phase flow as derived by Ishii (1975) and reviewed by Stuhmiller (1976). These equations formally describe a very wide class of possible flows of two phases, including the effects of mass and heat transfer, interfacial structure, compressibility, turbulence, and so on. For the purposes of studying the character of the model equations, however, we shall eliminate many of these effects by making simplifying assumptions which reduce this general formulation to one that is more manageable.

First, we shall neglect the effects of fluctuations. The proper description of turbulence in a multiphase flow is beyond the scope of the present work and its influence on the resulting model equations must be judged separately. Over a wide range of flows, however, the dominant *interfacial* stress is due to pressure and not turbulence. Second, we shall consider flows in which thermodynamic processes are unimportant, that is, those for which there is no heat or mass transfer between phases and for which both fluids may be separately considered as incompressible. In air–water mixtures, for example, this would be met in many engineering applications. Finally, we shall assume that viscous forces may be neglected in the bulk phases. Of course, at any surface in relative motion with respect to a fluid, such as the phase interface, a boundary layer must develop. For the blunt bodies we will consider, this thin, no-slip region profoundly modifies the pressure distribution along the surface no matter how small the viscosity. Above a critical Reynolds number, however, the interfacial forces are dominated by pressure, not the viscous stress itself. For spheres this critical value is about 24 (Batchelor 1967). Therefore, the effects of viscosity in the boundary layer will be incorporated indirectly through the modified interfacial pressure distribution. The macroscopic equations then reduce to

$$\frac{\partial \alpha_k}{\partial t} + \nabla \cdot (\alpha_k \mathbf{v}_k) = 0, \tag{1}$$

$$\alpha_k \rho_k \left\{ \frac{\partial \mathbf{v}_k}{\partial t} + (\mathbf{v}_k \cdot \nabla)\mathbf{v}_k \right\} = -\nabla(\alpha_k p_k) + \mathbf{M}_k, \tag{2}$$

$$\alpha_1 + \alpha_2 = 1, \tag{3}$$

where

$$\mathbf{M}_k = \sum_j \frac{-\mathbf{n}_k}{L_j} p_{ki}$$

is the interaction force due to the interfacial pressure. Here, $\mathbf{n}_k$ is the outward unit normal to the surface bounding phase k, $1/L_j$ is the interfacial area per unit volume, and the sum is understood to be over all interfacial elements within the averaging volume. Equations [1]–[3] are five equations for eight unknowns: α_k, $\mathbf{v}_k$, p_k, $\mathbf{M}_k$.

As can be seen, the interaction force is composed of a geometric part, $\mathbf{n}_k/L_j$, and a dynamic part, p_{ki}. For gas–liquid or liquid–liquid mixtures, an adequate description of the evolution of the interfacial geometric properties may well be the most challenging aspect of two-phase flow modeling. For now, however, we shall avoid these difficulties and assume that one phase, $k = 1$, is dispersed into many small, spherical regions, each of radius R. This greatly simplifies the calculation of the interfacial sums and, if a surface tension, σ, exists between the phases, then the balance of normal stresses at the interface requires

$$p_{1i} = p_{2i} + \frac{\sigma}{2R}. \qquad [4]$$

One final assumption concerns the bulk pressures. In the review by Stuhmiller (1976) it appeared that the closure assumption for the interfacial pressure and that for the bulk pressure play central roles. The present investigation focuses on the former, but to be specific, some choice of the latter must be made. Since the pressure in the dispersed phase is systematically higher by $\sigma/2R$ than that of the continuous phase, we shall take

$$p_1 = p_2 + \frac{\sigma}{2R}. \qquad [5]$$

This relation does not follow from the underlying conservation laws and deserves critical study in its own right, but since it has been employed explicitly in some models, e.g. Ramshaw & Trapp (1976) and tacitly in many others, it is a convenient bulk pressure closure relation for the present analysis.

With the help of the identity (Ishii 1975, p. 75)

$$\nabla \alpha_k = \sum_j \frac{-\mathbf{n}_k}{L_j} \qquad [6]$$

and the definitions $p_{2i} \equiv p_i$ and $p_2 \equiv p$, the one-dimensional form of the model equations become

$$\frac{\partial \alpha_1}{\partial t} + \frac{\partial}{\partial x}(\alpha_1 v_1) = 0, \qquad [7]$$

$$\frac{\partial \alpha_2}{\partial t} + \frac{\partial}{\partial x}(\alpha_2 v_2) = 0, \qquad [8]$$

$$\alpha_1 \rho_1 \left[\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x}\right] = -\frac{\partial}{\partial x}(\alpha_1 p) + M, \qquad [9]$$

$$\alpha_2 \rho_2 \left[\frac{\partial v_2}{\partial t} + v_2 \frac{\partial v_2}{\partial x}\right] = -\frac{\partial}{\partial x}(\alpha_2 p) - M, \qquad [10]$$

$$\alpha_1 + \alpha_2 = 1, \qquad [11]$$

$$M = \sum_j \frac{-n_1}{L_j} p_i. \qquad [12]$$

When the interfacial pressure distribution, p_i is given, [7]–[12] are six equations for six unknowns: α_1, α_2, v_1, v_2, p, M. We can now explore the consequences of various assumptions concerning that pressure distribution.

THE ASSUMPTION OF CONSTANT INTERFACIAL PRESSURE

The simplest constitutive relation for the interfacial pressure is that it is constant

$$p_i = \text{constant}, p^*. \qquad [13]$$

The interfacial force is then

$$M = p^* \frac{\partial \alpha_1}{\partial x}$$

and the model equations, written in matrix form, become

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & \rho_1\alpha_1 & 0 & 0 \\ 0 & 0 & \rho_2\alpha_2 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \alpha_1 \\ v_1 \\ v_2 \\ p \end{bmatrix} + \begin{bmatrix} v_1 & \alpha_1 & 0 & 0 \\ -v_2 & 0 & \alpha_2 & 0 \\ p - p^* & \rho_1\alpha_1 v_1 & 0 & \alpha_1 \\ p^* - p & 0 & \rho_2\alpha_2 v_2 & \alpha_2 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \alpha_1 \\ v_1 \\ v_2 \\ p \end{bmatrix} = 0. \qquad [14]$$

The characteristic values of this set satisfy the equation

$$\begin{Vmatrix} \lambda + v_1 & \alpha_1 & 0 & 0 \\ -(\lambda + v_2) & 0 & \alpha_2 & 0 \\ p - p^* & \rho_1\alpha_1(\lambda + v_1) & 0 & \alpha_1 \\ p^* - p & 0 & \rho_2\alpha_2(\lambda + v_2) & \alpha_2 \end{Vmatrix} = 0,$$

which, when expanded, is

$$p - p^* = \alpha_2\rho_1(\lambda + v_1)^2 + \alpha_1\rho_2(\lambda + v_2)^2.$$

The condition that the characteristic values of the equation

$$A(\lambda + v_1)^2 + 2B(\lambda + v_1)(\lambda + v_2) + C(\lambda + v_2)^2 = D(v_1 - v_2)^2$$

be real is that the discriminant is positive, that is,

$$B^2 + (A + 2B + C)D \geq AC.$$

In the present case this becomes

$$(\alpha_2\rho_1 + \alpha_1\rho_2)(p - p^*) \geq \alpha_1\alpha_2\rho_1\rho_2(v_1 - v_2)^2. \qquad [15]$$

If we were to make the additional assumption that $p = p^*$, an assumption which leads to the single pressure model proposed by several authors, then the only case for which real characteristics exist is the homogeneous flow, $v_1 = v_2$. For the more important case of unequal velocities, [15] indicates the interface pressure must be systematically lower than the bulk pressure to retain real characteristics, in fact, lower by an amount proportional to the square of the relative velocity. Since by Bernoulli's law, the pressure *does* vary as the square of the velocity along a streamline, and is systematically *lower* than the free stream value in regions of high velocity, such as occurs in the neighborhood of an obstruction, then one might anticipate that the inclusion of more details of the subscale flow could lead to equations with real characteristics.

Even with the resolution of the complex characteristic behavior, however, the ρ_i = constant equations have the aphysical property that for a given applied pressure gradient the phases accelerate inversely as their densities. Thus a bubble of air released in a glass of water would accelerate at 1000 g! Not only is this phenomenon not observed, but the cause of the dilemma has been well studied. Such a description neglects the inertia of the surrounding, heavier fluid which must be moved to allow the bubble to rise. The bubble acts as though it has an extra inertia comparable to that of the surrounding fluid—the so-called added mass effect. This is another property which arises out of the influence of the microscale flow and is lost in the p_i = constant assumption. Before considering more realistic pressure distributions, a comment on characteristics is in order.

COMPLEX CHARACTERISTICS AND SOLUTION BOUNDEDNESS

For *linear*, partial differential equations, the occurrence of complex characteristics is associated with several aphysical properties, including the exponential growth of disturbances, especially at small space and time scales. This is illustrated by the equation set

$$\frac{\partial f}{\partial t}+\frac{\partial h}{\partial x}=0$$

$$\frac{\partial h}{\partial t}-\frac{\partial f}{\partial x}=0,$$

whose characteristic values are $\lambda = \pm i$. The evolution of a local disturbance in an infinite domain, or one for which f or $\partial f/\partial x$ vanishes at the end points, can be easily shown to satisfy

$$\frac{\partial^2}{\partial t^2}\int \frac{1}{2} f^2\,\mathrm{d}x = \int \left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial t}\right)^2\right\}\mathrm{d}x > 0.$$

Thus, disturbances grow without bound and the most rapidly growing are those whose spatial or temporal scales are small.

For *quasi-linear*, partial differential equations the conclusions are less obvious. From [14] one can derive the mechanical energy variation

$$\frac{\partial}{\partial t}\int \sum_{k=1}^{2}\frac{1}{2}\alpha_k\rho_k v_k^2\,\mathrm{d}x = \int (p^*-p)\sum_{k=1}^{2} v_k\frac{\partial \alpha_k}{\partial x}\,\mathrm{d}x.$$

When the velocities are unequal and $p = p^*$, the characteristic values are complex, yet *the solution is bounded and the mechanical energy conserved for all wavelength disturbances.* This is one indication that the theorems and intuition that have developed regarding linear pde's with complex characteristics may not apply to quasi-linear pde's without some modification.

FLOW ABOUT A SPHERE

We imagine a situation in which there are many dispersed particles within the averaging volume, precisely the limit in which one would expect the averaging concepts to be most meaningful. When the individual particles are widely enough spaced, that is for small enough volume fraction of the dispersed phase, the flow about each can be considered to be unaffected by neighbors. We will relax this condition later. For *inviscid* flow about a sphere in arbitrary motion the pressure distribution along the surface is given by Lamb (1932) as

$$p_i = p + \frac{1}{2}\rho_2 R\frac{\mathrm{d}U}{\mathrm{d}t}\cos\theta + \frac{1}{8}\rho_2 U^2(9\cos^2\theta - 5),$$

where U is the instantaneous relative velocity, $v_1 - v_2$, θ is the polar angle measured with respect to the direction of U, and the time derivative is made following the dispersed particle, that is, $\mathrm{d}/\mathrm{d}t = \partial/\partial t + v_1(\partial/\partial x)$.

Experimental observation has confirmed that the effects of acceleration are adequately described by the above expression (c.f. Weigel 1964, Chap. 11), but for a Reynolds number greater than about 24, flow separation destroys the fore-aft symmetry in the non-accelerated component of the pressure (Batchelor 1967, p. 325 ff). Since for $Re < 24$, the net interfacial force is primarily due to viscous shear stress, which we have neglected in our analysis, we shall consider the higher Re range. Only the most gentle motions fall in the low Reynolds number range; for example a 1 cm bubble moving at 1 cm per sec through water corresponds to $Re = 100$. Consistent with this range of parameters, then, the pressure distribution about a sphere must be generalized to

$$p_i = p + \frac{1}{2}\rho_2 R \frac{\mathrm{d}U}{\mathrm{d}t}\cos\theta + \rho_2 U^2 F(\theta), \qquad [16]$$

where $F(\theta)$ is determined experimentally. We can now proceed to evaluate the interfacial force M.

The explicit evaluation of the interfacial sum depends somewhat on the distribution properties of the dispersed particles within the averaging volume: the number contained entirely within the volume, the number intersecting a particular face of the averaging volume with a particular inclusion angle, etc. We can obtain a first-order estimate, however, by using the sum identity [6] and appealing to the large number of particles within the averaging volume. We begin by separating out the average interfacial pressure level.

$$p_i = \langle p_i\rangle + p_i',$$

$$\langle p_i\rangle = p + \rho_2 U^2\langle F(\theta)\rangle,$$

$$p_i' = \frac{1}{2}\rho_2 R\,(\mathrm{d}U/\mathrm{d}t)\cos\theta + \rho_2 U^2 F'(\theta),$$

where $\langle\,\rangle$ represent an average over the surface of a sphere and $F' = F - \langle F\rangle$. Of course, $\langle\cos\theta\rangle = 0$. The mean part is readily evaluated, using [6], as

$$\sum_j \frac{-n_1}{L_j}\langle p_i\rangle = \langle p_i\rangle\frac{\partial\alpha_1}{\partial x} = (p + \langle F(\theta)\rangle\rho_2 U^2)\frac{\partial\alpha_1}{\partial x}.$$

For the pressure distribution quoted in Schlichting (1959) and shown in figure 1, we have $\langle F(\theta)\rangle = -0.37C_D$.

The varying part makes a non-zero contribution to the particles completely contained within the volume (the constant part does not) and, for an averaging volume large compared to a particle volume, the vast majority of particles are entirely within the volume. Since the limits of integration are free of distributional considerations in this case, the integration can be readily carried out. Specifically,

$$\sum_j \frac{-n_j}{L_j}p_i' = \sum_{\text{particles}}\frac{1}{V_{\text{av}}}\int_{-\pi}^{\pi} R\,d\phi\int_0^{\pi} R\sin\theta\,\mathrm{d}\theta(-\cos\theta)\left\{\frac{1}{2}\rho_2 R\frac{\mathrm{d}U}{\mathrm{d}t}\cos\theta + \rho_2 U^2 F'(\theta)\right\}.$$

The integral of the first term gives

$$\sum_{\text{particles}} -\frac{1}{2}\rho_2\frac{\mathrm{d}U}{\mathrm{d}t}\frac{\frac{4}{3}\pi R^3}{V_{\text{av}}} = -\frac{1}{2}\alpha_1\rho_2\frac{\mathrm{d}U}{\mathrm{d}t}.$$

The integral of the second term is nothing more than the net force exerted on that sphere due to pressure forces, that is, the form drag. In terms of a drag coefficient, we have

$$\sum_{\text{particles}}\frac{1}{V_{\text{av}}}C_D\frac{1}{2}\rho U^2\pi R^2 = \pm\frac{1}{8}C_D\rho_2 U^2\frac{1}{L_s}.$$

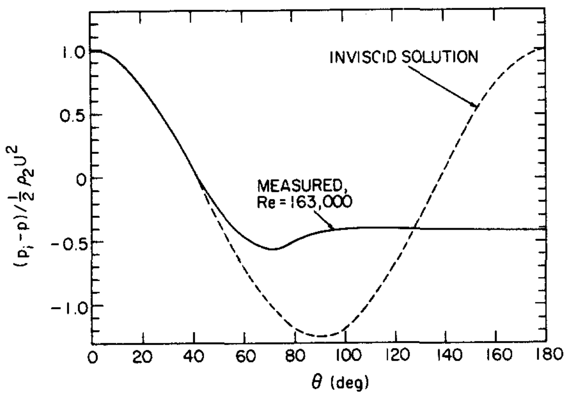

Figure 1. Pressure distribution about a sphere.

The sign of this term is understood to be such as to oppose the relative motion. Here $1/L_s = \rightarrow 1/L_j$ is the total interfacial area per unit volume; we have also made use of the particular relation between the projected area and total area of a sphere.

Combining these results we have

$$M = (p - 0.37C_D\rho_2U^2)\frac{\partial\alpha_1}{\partial x} - \frac{1}{2}\alpha_1\rho_2\frac{dU}{dt} \pm \frac{1}{8}C_D\rho_2U^2\frac{1}{L_s}. \quad [17]$$

It seems appropriate at this time to define three dimensionless numbers: the added mass coefficient, μ; the dynamic-pressure coefficient, ξ; and the drag parameter, η; allowing us to write [17] as

$$M = (p - \xi\rho_2U^2)\frac{\partial\alpha_1}{\partial x} - \mu\alpha_1\rho_2\frac{dU}{dt} \pm \eta\frac{\rho_2U^2}{L_s}. \quad [18]$$

This latter form not only makes the mathematical manipulations that follow less cumbersome, but it identifies in parameter form the three effects resulting from the consideration of the interfacial pressure distribution.

The model equations are now

$$\frac{\partial\alpha_1}{\partial t} + \frac{\partial}{\partial x}(\alpha_1v_1) = 0, \quad [19]$$

$$\frac{\partial\alpha_2}{\partial t} + \frac{\partial}{\partial x}(\alpha_2v_2) = 0, \quad [20]$$

$$\alpha_1(\rho_1 + \mu\rho_2)\left\{\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x}\right\} - \alpha_1\mu\rho_2\left\{\frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial x}\right\} = -\alpha_1\frac{\partial p}{\partial x} - \xi\rho_2(v_1 - v_2)^2\frac{\partial\alpha_1}{\partial x} \pm \eta\frac{1}{L_s}\rho_2U^2 \quad [21]$$

$$(\alpha_2 + \mu\alpha_1)\rho_2\frac{\partial v_2}{\partial t} + (\alpha_2v_2 + \mu\alpha_1v_1)\rho_2\frac{\partial v_2}{\partial x} - \alpha_1\mu\rho_2\left\{\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x}\right\}$$
$$= -\alpha_2\frac{\partial p}{\partial x} + \xi\rho_2(v_1 - v_2)^2\frac{\partial\alpha_1}{\partial x} \mp \eta\frac{1}{L_s}\rho_2U^2. \quad [22]$$

The characteristic values satisfy the equation

$$\left\{\alpha_2\frac{\rho_1}{\rho_2} + \mu\right\}(\lambda + v_1)^2 + \left\{\frac{\alpha_1}{\alpha_2}\mu\right\}(\lambda + v_1)(\lambda + v_2) + \alpha_1(\lambda + v_2)^2 = \xi(v_1 - v_2)^2 \quad [23]$$

which leads to the condition

$$\frac{1}{4}\left\{\frac{\alpha_1}{\alpha_2}\mu\right\}^2+\left\{\alpha_2\frac{\rho_1}{\rho_2}+\mu+\alpha_1+\frac{\alpha_1}{\alpha_2}\mu\right\}\xi\geq\left\{\alpha_2\frac{\rho_1}{\rho_2}+\mu\right\}\alpha_1 \quad [24]$$

for real roots. Clearly, this inequality will be met if

$$\xi\geq\alpha_1, \quad [25]$$

for any density ratio and any added mass coefficient. Since $\xi\geq 0.17$ for a single, spherical particle, that is, in the limit as $\alpha_1\rightarrow 0$, we can see that *the equation set* [19]–[22] *has real characteristics in those flow regimes for which the microscale pressure distribution used is valid.*

Furthermore, the added mass behavior, which removes the aphysical over-acceleration of light masses, and the form drag interaction forces, that provide the limit on the allowed slip velocity, appear as natural consequences of the inclusion of the variability of the interfacial pressure distribution. Although the evaluation of the interfacial sums is admittedly primitive in the present treatment, it does indicate that all of these very real and important effects can be systematically accounted for in a macroscale model.

EXTENSION TO PARTICLE–PARTICLE INTERACTIONS AND OTHER INTERFACIAL GEOMETRIES

The previous results demonstrate that ambiguities can be removed at small volume fractions, $\alpha\lesssim 0.17$. Even at this apparently small value, however, the surfaces of neighboring particles in a cubic-packed array are less than a diameter apart, so that one might expect particle–particle interactions to be of some importance. One of the effects to be expected is that particles will be partially immersed in neighboring wakes, lowering the bulk-averaged velocity difference. Thus the drag coefficient, based on the macroscopic velocity difference, must increase rapidly with the volume fraction. This is indeed borne out in experimental measurements of flow about solid spheres conducted by Rowe and quoted by Wallis (1967), pp. 178 ff, where the drag coefficient obeys the law

$$C_D(\alpha_1)=C_D(0)(1-\alpha_1)^{-2.7}. \quad [26]$$

For $\alpha_1=0.17$, the drag coefficient has increased by a factor of 1.65 and the range of parameters for which [25] is met is greatly increased. In fact, if we use [26] in [25] we obtain

$$0.17\geq\alpha_1(1-\alpha_1)^{2.7}. \quad (27)$$

The maximum value of the R.H.S. of [27] is only 0.12 so that we find that *the equation set* [19]–[22] *has real characteristics for all volume fractions when particle–particle interactions are taken into account.* Of course, with the close packing that occurs even for $\alpha_1<0.17$, region coalescence and interfacial distortions will probably play an even greater role in fluid–fluid flows.

Much of the previous work can be taken over to non-spherical regions with only slight modification. As long as interfacial distortions are not allowed, surface tension still plays no role; the added mass and drag coefficients can, in principle, be measured or calculated for arbitrarily shaped regions. For example, Lamb (1932), p. 155, tabulates the added mass coefficients of an ellipsoid for various degrees of elongation and orientation; in general end-on motion results in $0\leq\mu\leq 0.5$, while broadside motion gives $0.5\leq\mu\leq 1.0$.

For streamlined shapes, the dynamic-pressure coefficient ξ is no longer principally related to the drag coefficient, since the flow separation region is confined to a small area near the back and most of the drag is due to viscous stresses in the turbulent boundary layer. The low pressure area brought about by the increase in flow velocity near the body, however, extends over the entire surface, with a negligible high pressure area near the front. For the pressure distribution, shown in figure 2, measured about a long body of revolution (Schlichting 1959, p. 20) we find $\xi = 0.06$, so that, even with the neglect of the turbulent shear stress, the interfacial pressure forces make a large contribution toward keeping the characteristics real. For these complex, extended bodies where many more degrees of freedom are allowed, the present analysis is probably inadequate. The complete description will require, then, incorporating more of the effects of the microscale processes and the dynamic evolution of the interface.

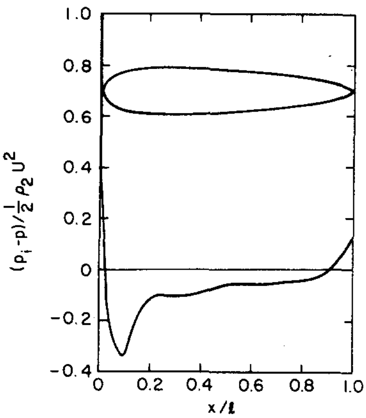

Figure 2. Pressure distribution about an axisymmetric streamlined body.

SUMMARY

The modeling of two-phase flows is certainly in a very formative stage and one cannot be sure that all of the important questions have been raised, let alone that any have been answered. It is difficult to foresee which will emerge as the most pertinent issues and how they will be resolved. In the meantime, heuristic reasoning will play a central role in such investigations.

The present work is no exception, despite its attempt to make explicit assumptions and approximations and to begin with an exact formulation. The neglect of terms arising from fluctuations and the use of a bulk pressure closure relation may influence our conclusions in some profound way that is not now evident.

In spite of the uncertainties, however, progress appears to have been made. The results presented show that including information about the subscale flow can drastically alter the macroscale model and that when the information is reasonably complete this alteration seems to be in the right direction, that is, aphysical behavior is removed in a natural way. The model equations [19]–[22] contain most of the effects that one would expect of the dispersed flow regime and might be expected to be an improvement on many of the elementary two-phase flow models.

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